

Galileo's Mathematical Language of Nature¹

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The great book of Nature lies ever open before our eyes and the true philosophy is written in it . . . But we cannot read it unless we have first learned the language and the characters in which it is written . . . It is written in mathematical language and the characters are triangles, circles, and other geometrical figures. (Galileo 1623, p. 232)

ABSTRACT. Undergraduate students do not always make a clear distinction between physics and mathematics, particularly early in their studies. We offer a simple historical example and show how it can be used to illustrate some of the important differences and relationships between the two. The example is Galileo's treatment of motion under uniform acceleration, in which he uses geometry instead of algebra to represent quantities such as time and velocity and stresses the need to test the adequacy of the representation by experiment. The general importance of Galileo's work in the history of science and the fact that it is accessible to undergraduates not concentrating their studies in mathematics or the sciences make it particularly suitable for our purposes. In addition to undergraduate courses in physics or mathematics, many of the points we make should be useful in courses in the history and philosophy of science and mathematics.

INTRODUCTION

Beginning physics students sometimes confuse the mathematics used to do physics with the physics itself. (The Zen aphorism 'when someone points at the Moon, you do not look at his finger' comes to mind.) We will begin with a look at Galileo's solution of the problem of motion under uniform acceleration (Galileo 1638, p. 206). Then we will show how his discussion can be used to provide students with helpful illustrations of at least three interesting points about the relationship between mathematics and physics. First, Galileo's well known insistence upon experiment as crucial for testing the adequacy of his geometrical representations, coming as it does after he has completed his proofs, can be used to illustrate the fact that claims in physics are not identical with claims in mathematics. Second, beginning students are typically introduced to Galileo's kinematics by means of geometrical representations such as $v-t$ diagrams and required to shift to algebraic representations for purposes of calculating results and displaying relationships between his kinematics and more general mechanical principles; and looking at Galileo's own way of representing physical quantities can provide a vivid and historically important demonstration that the physical laws and theories need not be uniquely correlated with any *particular* mathematical representations. Fin-

ally, a discussion of why, three hundred and fifty years ago, it was appropriate for Galileo to use geometry to represent his claims about motion in his arguments, while now it is best to use algebra for such purposes, can provide a good illustration of two points: that one mathematical representation can be better than another,² even where both fit with experiment and observation; and that, as mathematics provides new tools to physics, the best ways mathematically to represent physical phenomena can change.

MOTION UNDER UNIFORM ACCELERATION ACCORDING TO GALILEO

Galileo argues that natural acceleration (the acceleration of falling bodies) should be identified with uniform acceleration from rest. After arguing against the Aristotelian idea that this uniform acceleration is to be understood as occurring when velocity receives increases proportional to the distance traversed,³ he states that a body's motion is uniformly accelerated when, starting from rest, its 'momentum' receives equal increments in equal times.⁴ He then proves that, if uniform acceleration is defined in this way, his Theorem I, Proposition I follows:

Theorem I, Proposition I: The time in which any space is traversed by a body starting from rest and uniformly accelerated is equal to the time in which that same space would be traversed by the same body moving at a uniform speed whose value is the mean of the highest speed and the speed just before the acceleration began. (Galileo 1638, p. 205)

Galileo begins his proof by drawing the construction in Figure 1. The segment CD represents the distance traveled during uniformly accelerated motion starting from rest at C . The segment AB is chosen to represent the time taken during the trip CD , and the segment EB represents the final speed at the end of the fall. Galileo next calls for a line to be drawn from A to E . Points on the segment AB represent different times during the fall, and segments starting at those points and drawn between the line AE and AB are proportional to the speed at that time. He completes his construction with the segment GF where F is a point which bisects EB and GF is parallel to AB . Segments between AB and GF are speeds which correspond to an object moving with uniform velocity (no acceleration).

We note that area AGI is equal to area IEF , since they are similar triangles with the same base ($AG = EF$). And this implies that $AGFB$ is the same area as AEB . If we take area in this diagram to represent distance traveled, then $AGFB$ represents the distance traveled in the constant speed case and AEB represents the distance traveled in the accelerated case; and the fact that $AGFB = AEB$ proves that the distance traveled in a given time with uniform acceleration equals the distance traveled with constant speed, if the constant speed (GA) is one half the maximum speed (EB) of the accelerated case.

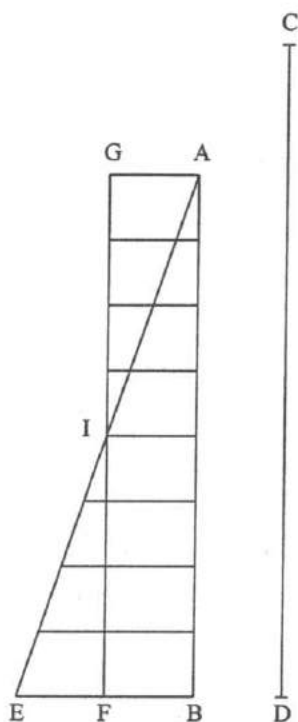


Figure 1.

However, Galileo's argument does not proceed in this way because he needs to make the connection between the effects of uniform speed and those of uniform acceleration in terms of velocities at instants. To do this he equates the area of each of the figures to the 'sum of all the parallels' contained in it, arguing that 'the parallelogram $AGFB$ will be equal in area to the triangle AEB , since . . . the sum of all the parallels contained in the quadrilateral is equal to the sum of those contained in the triangle'. The principle he uses here is similar to that in Cavalieri's theorem,⁶ familiar from some calculus texts: two solids have the same volume if corresponding cross-sections have the same area. In the present case, Galileo argues that the sum of all the parallels in IEF is the same as the sum of those in GIA , and the sum of the parallels in $AIFB$ is common to both figures. All the parallels represent the velocities at all the instants of time during the time interval, and so the sum of them all will represent the sum of all the momenta (the quantity of motion) of the moving body; and since this sum is the same in each case, the distance traveled must be the same.

Galileo has used geometric properties of his construction to prove quantitative relationships among elements of the construction which, he claims, represent corresponding quantitative relationships among the physical quantities of time, velocity, 'momentum', and distance. By modern standards he is not always clear in his mathematical reasoning.⁷ But a little historical imagination makes it hard to fault him for this since he worked thirty to fifty years before Newton, and two hundred years before Cauchy's definition of a limit; moreover it is possible to correct the proof in a reasonable way with the help of mathematical tools which have been developed since Galileo's time.⁸

THE MATHEMATICS AND THE PHYSICS DISTINGUISHED

Galileo makes it clear that to grant him the geometrical properties of his construction does not commit his reader to granting him that the corresponding quantitative relationships among the physical quantities of time, velocity, 'momentum', distance and natural acceleration hold. After proving another proposition he states that experiments need to be cited to show that uniform acceleration as he represents it is indeed 'that which one meets in nature in the case of falling bodies'. It is helpful to ask students to consider a question here: If he has already proven what he set out to prove, exactly why is experiment needed?⁹ Careful discussion of this question is, we think, an excellent way to bring students to understand that Galileo is right about the need for experiment because it is one thing to prove the mathematical relationships, and another thing to demonstrate that the physical relationships correspond.¹⁰ His proof draws out some of the features of his mathematical map, but to do physics he needs to show the applicability of his mathematical map to the physical world: the mathematics itself isn't physics. Thus his general statement that 'in those sciences where mathematical demonstration are applied to natural phenomena . . . The principles, *once established by well-chosen experiments* [our emphasis], become the foundations of the entire superstructure'.

TWO MATHEMATICAL MAPS

Galileo's construction represents a special case of a relation easily recognized by students in its usual algebraic representation: $s = v_{ave}t$, where s is the distance, t is time, and v_{ave} is the average velocity and equals $(v + v_0)/2$ with v being the final speed and v_0 the initial speed. When the initial velocity is $v_0 = 0$ we have $s = \frac{1}{2}vt$, which is the usual algebraic representation of what Galileo represents with Figure 1. However, Galileo's representation in Figure 1 is no less accurate than the algebraic one.

This point clearly shows that to identify velocities, times, distances and accelerations with numbers is no more (or less) plausible than to identify distance with areas or time intervals or velocities with line segments.

HOW ONE OF TWO ACCURATE MATHEMATICAL MAPS CAN BE BETTER THAN THE OTHER

When one has convinced the student that Galileo's geometrical representation is as accurate as the corresponding algebraic one, it is helpful to consider some of the reasons why it was reasonable for him to represent the physical relationships the way he did, rather than using algebra as we would do now. And, on the other hand, why we ought to stick with the algebraic representation. The explanation we would offer appeals to the state of mathematics in Galileo's day and how it has changed since. Thus it requires something like the following history lesson.

Mathematical researches during Galileo's time¹² largely focused on geometry. In Italy, there seem to have been two trends. One trend, of which Galileo is representative, concentrated on a revival and expansion of ancient Greek geometry. Galileo was particularly fond of Euclid and Archimedes, and he sought in his own work to extend the application of their techniques to new areas. The other trend was busy working to develop geometry by the use and development of algebra as imported from the Arabic world. This movement began in Italy¹³ and eventually spread to France and England.

Although significant work on algebra dated to the mid 1500s,¹⁴ there are good reasons for thinking that it was insufficiently developed before the mid 1600s to be useful as a reliable tool for physical investigations. For one thing, the notation was not standardized and still quite rough.¹⁵ But a more serious problem was that the theoretical basis for it was still poorly understood. The modern notion of algebra taken as axioms of a logical system, so that identities can be proven by formal manipulation, is a relatively recent idea. In the late 16th and early 17th centuries algebra was parasitic on pre-algebraic geometry: algebraic statements were thought of in geometric terms, and algebraic rules were proven using pre-existing geometrical axiom systems. For example, $(a + b)^2 = a^2 + 2ab + b^2$ would be understood as a statement about how a square $a + b$ on a side could be divided into two squares and two rectangles. Vieta and Descartes began the use of algebra to solve geometrical problems, and Descartes appears to have been the first to realize that an expression such as a^2 can indicate the length of a line segment as well as an area.

A related difficulty from a modern perspective was that numbers, particularly irrational numbers, were not well understood, and had tended themselves to be thought of in pre-algebraic geometrical terms ever since Pythagoras: thus the square root of 2 was thought of as the diagonal of a unit square. It had been known since antiquity that not all numbers

could be represented as the ratio of two integers, and there was no well-developed system of notation to handle irrationalities – although they could be approximated with great precision it was difficult to think of them except in geometrical terms.

One of the results of all this is the concept of magnitude shared by Galileo and others of his time. Today we associate magnitude with numbers, independently of any particular geometrical representation; likewise ratios and irrational numbers. Galileo, however, associates magnitudes and relations among them with geometrical properties. Given the state of mathematics in his day, Galileo might reasonably claim to have used the best tool available to him – pre-algebraic geometry. From this perspective, as he put it in *Il Saggiatore*, the characters in the ‘language of mathematics . . . are triangles, circles, and other geometrical figures’ (Galileo 1623).

Of course, given the state of mathematics in *our* day, we are, similarly, under an obligation to use the best tool available to *us*; and that is the algebraic representation: $s = \frac{1}{2}vt$, rather than Figure 1. But (question to raise for students to consider) we have already granted that *both* representations are accurate, so what justifies the claim that our way is *better* than Galileo’s way? The goal is to get students to realize that there are at least two purposes for which the algebra is better: one would be that for calculational purposes the algebra is usually simpler to use;¹⁶ another would be that the algebraic representation is embedded in a more powerful system so that more mathematical consequences can be drawn when we use the resources of algebra. If we were leading the discussion we would ask students to struggle with examples such as the following:

Galileo’s construction in Figure 1 corresponds to the algebraic representation $s = \frac{1}{2}vt$, which is easily derived from the general case represented algebraically as we indicate in section 4, above: $s = v_{ave}t$, where s is the distance, t is time, and v_{ave} is the average velocity and equals $(v + v_0)/2$ with v being the final speed and v_0 the initial speed. To prove it, we simply set the initial velocity $v_0 = 0$. We ask the student to represent the general case (including cases of negative initial velocities) using the pre-algebraic methods of representation and proof which Galileo used to represent the special case.

NOTES

¹ We wish to thank an anonymous reviewer for *Science & Education* for helpful critical remarks on an earlier version of this paper.

² Unless we explicitly state otherwise, when we claim that one mathematical representation of a purported law or theory is ‘better’ than another we mean that it is better for purposes of displaying its connections with experimental evidence and with other laws and theories.

³ Here we have in mind primarily the late mediaeval Aristotelians who associated natural acceleration with $\Delta v/\Delta s$. Aristotle himself had argued (*Physics* 265b10–15) that the natural rectilinear motion of bodies toward the center of the earth is non-uniform from the starting point and toward the center, since “the farther they are from the state of rest, the faster they travel”. The mediaevals followed him in this respect, and added the assumption that

there is a simple positive correlation between Δv and Δs in natural acceleration. For a good discussion of this, see Copleston.

⁴ Galileo's notion of momentum (Galileo 1638, p. 200) does not correspond to our current notion. It is a concept of the natural 'quantity of motion', developed out of work by Jean Buridan and Nicolas of Oresme. Momentum is proportional to how much stuff is moving and how fast it is moving. Thus, when the amount of stuff is constant (as in the case Galileo is considering) changes in the 'momentum' are directly proportional to the velocity. For that reason Galileo describes his definition as 'the same as saying that in equal time-intervals the body receives equal increments of velocity'.

⁵ Galileo has previously defined a uniform motion as 'one in which the distances traversed by the moving body during any equal intervals of time, are themselves equal' (Galileo 1638, p. 197).

⁶ Cavalieri was a student of Galileo.

⁷ In addition to the difficulties we mention in note 8, below, an anonymous reviewer makes the interesting point that, accepting Galileo's proof that if uniform acceleration (as he defines it) occurs, then $s = \frac{1}{2}vt$ for that occurrence, he does not prove the converse (that every case in which $s = \frac{1}{2}vt$ is a case of uniform acceleration) although that is implicitly assumed in his experimental verification.

⁸ In Galileo's work we find the germ of the idea of infinitesimals; and it is worth mentioning to students, at least in passing, some of the very serious problems that this idea presented. Difficulties had been present since ancient times. The classic formulations of them are the familiar paradoxes of Zeno. An example is the Achilles paradox: Achilles cannot overtake a tortoise in a race, for as soon as he runs to a point where the tortoise was, the tortoise has advanced to a point further along the track. In the *Two New Sciences*, Galileo uses an argument from Aristotle's *Physics* to deal with a variant of this paradox: the infinite series of runs can be completed because there is an infinite series of time intervals available for its completion. Sometimes it is claimed that this and other paradoxes presented by Zeno are resolved by the use of the idea of an infinite convergent series. However, students who take this line of reasoning can be shown that difficulties remain, by citing another of Zeno's paradoxes known as the Plurality: suppose a spatial distance is thought of as consisting of an infinite number of indivisible points. If each point has finite length, then the distance is infinite in extent; but if each point (instant) has zero length, then the distance is zero. This way of looking at distances can be applied to the situation of Achilles and the tortoise by querying the size of his last run: if it is of finite positive length, then each of the infinite number of runs preceding it is larger than it, and the distance Achilles must cover before making the last run is infinite in extent; but if his last run is of zero size, it is not the last run since he has already caught the tortoise; and if there is no last run (no run at the termination of which he has caught up with the tortoise) then he does not complete the series and so does not catch the tortoise. A modern solution to the paradoxes which avoids this difficulty has been offered by Adolph Grunbaum (Grunbaum 1967, esp. chapters 2 & 3; 1969). But Grunbaum's solution uses heavy equipment unavailable to Galileo or Newton: superdenumerable infinite aggregates, modern topology, and measure theory.

⁹ It is important to separate two different questions here. First, if we grant Galileo his mathematical proofs, why is it necessary to verify that $s = \frac{1}{2}vt$ by experiment *at all*? A clear answer to this question depends upon a clear distinction between mathematics and physics. Second, once it is clear that experimental verification is necessary, *how* can one do that? Galileo's handling of the problem of designing an experiment to verify that $s = \frac{1}{2}vt$, when he had no way of measuring the velocity directly, is worth discussing as illustrating some of the difficulties involved in working out ways to test hypotheses by observation. But discussion of the first question is less common in introductory physics texts, and that is the one we refer to here.

¹⁰ For mathematics to be applicable to the natural world, not only numbers but also the formal properties of the symbolic operations of mathematics must represent physical properties and physical operations. During this century, the investigation of the properties which physical magnitudes must have in order to be representable and systematically elaborated

by the use of mathematical representations has come to be known as measurement theory. For full discussion of this point, and measurement theory in general, see, eg., Nagel (1932) and Suppes (1951).

¹¹ A more recent example of two different mathematical maps would be Schrödinger's wave mechanics and Heisenberg's matrix mechanics. Both are accurate representations of quantum mechanics.

¹² Galileo lived from 1564 to 1642. As a young man, following the wishes of his father, he attended the University at Pisa to study medicine. However, he soon showed an interest in mathematics and eventually he left the University without receiving a degree. After a period of time in which he tutored privately in mathematics Galileo returned to the University of Pisa as a member of the faculty; and during his stay there he did much of his work on falling bodies. In 1592 he was appointed professor of mathematics at the University of Padua for a term of six years which was eventually extended to a lifetime position. *The Two New Sciences* appeared near the end of his life, in 1638 (Geymonat 1965; McMullin 1967; Kline 1972).

¹³ In the decades before Galileo's move to Padua, the University there was a leading center for the development of algebra.

¹⁴ Cardan's *Ars Magna*, which contained the first published solution of the cubic polynomial equation, appeared in 1545, and Vieta's *Zeteticoum Libri Quinque* in 1593.

¹⁵ Vieta wrote $a^3 + 3a^2b + 3ab^2 + b^3 = (a + b)^3$ as *a cubus + b in a quadr. 3 + a in b quadr. 3 + b cubo aequalia a + b cubo*. Descartes was the first to use letters at the end of the alphabet to refer to unknowns in his work on geometry published in 1637 as an appendix to his *Discourse on Method*.

¹⁶ The example of Schrödinger's wave mechanics and Heisenberg's matrix mechanics (see note 11, above) provides an illustration of this practical aspect of mathematical representations: which system a physicist uses will depend upon the quantum mechanical problems being addressed.

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