

cal value. We hope that it will find recognition and contribute toward making density-functional theory more accessible to students.

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Comment on "Time-frequency analysis with the continuous wavelet transform," by W. Christopher Lang and Kyle Forinash [*Am. J. Phys.* **66** (9), 794–797 (1998)]

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We read with much interest the paper by Lang and Forinash (Ref. 1) but felt we needed to add words of caution regarding wavelet analysis. We state at the outset, though, it is not our intent to either blemish or deny the results of Lang and Forinash. We do, however, want to make clear that, when applied to well-defined problems, *wavelets work wonders*. When applied, on the other hand, to less well-defined problems, the wavelet transform *does not always* "produce spectrograms which show the frequency content of sounds (or other signals) as a function of time in a manner analogous to sheet music."

The analysis in Ref. 1 is indeed accurate and to the point and produces the desired results in the cases described there because the signals analyzed were artificially created; consequently, their pitch and frequency content were known *a priori*. In phenomena where intensity and frequency content are not known *a priori*, such as turbulent flows and other random signals, the wavelet transform often obscures significant information in the signal. As an example, a wavelet that is a second-difference operator, such as the French-hat wavelet, can provide no information on the *linear trend* in a signal. For the unfamiliar reader, the French-hat wavelet is defined as $\{-B(3\alpha) + 2B(3\alpha - 1) - B(3\alpha - 2)\}/2\sqrt{L}$, where $\alpha = t/L$ and $B(\alpha)$ is the standard box function. That is, $B(\alpha) = 1$ for $0 \leq \alpha \leq 1$, and $B(\alpha) = 0$ otherwise.

The wavelets problem for practitioners studying random turbulence is much like the problem posed by the professor in a chemistry class when she gives a beaker of liquid to a group of students and asks them to identify its contents. The students conduct a variety of tests which are known to reveal specific elements or compounds and then report their results. If the liquid contains elements for which there are no known tests, or if the students neglect to conduct a certain test, the students cannot identify all the contents. The same problem exists in identifying computer viruses—you can identify only the *known* ones.

In Ref. 2 we demonstrate that wavelet analysis has limitations which are not widely appreciated; failure to recognize these can lead to misinterpretations. Reference 1 correctly points out the limitations of both the short-time Fourier transform and the Gabor transform, as well as the limitations the *uncertainty principle* imposes on both the Fourier transform and the wavelet transform. But the limitations on wave-

let analysis we emphasize in Ref. 2 are just as fundamental. There we applied wavelet analysis to nonstationary turbulence data, but our results apply to any random signal in general. Our main point is that a given signal may contain components that are orthogonal to the analysis (*mother*) wavelet; consequently, for a wavelet analysis to be viable, the analysis wavelet must be carefully matched to the phenomenon of interest. That is, you must have some *a priori* idea as to what scale elements are present in the signal and which wavelets are best suited for isolating them.

Moreover, most wavelets are symmetric about the localization time (usually denoted as t_0) and therefore assign the same weight to those elements of the signal *forward* in time from t_0 by an amount τ as they do to those elements *backward* in time from t_0 by an equal amount. In phenomena such as turbulence, where energy dissipation and its companion irreversibility are commonplace, such an assignment is plausible when τ is small but cannot be a reliable characterization of the behavior when τ is large; see Refs. 3, 4.

One of the limitations Lang and Forinash (Ref. 1) point out with respect to the Wigner distribution $V\{f(t, \omega)\}$ —that a spectrogram based on $V\{f(t, \omega)\}$ will show interference artifacts or noise in regions where none should be—is not really a limitation at all. In their analysis it does produce the artifacts indicated, but in turbulent signals such artifacts are nonlinearities that appear routinely in signals encountered in nature. The source of the confusion is that $V\{f(t, \omega)\}$ is denoted as the *Wigner distribution of f*, the implication being that frequencies revealed by $V\{f(t, \omega)\}$ are the frequencies prevailing in f . This is true only if f is a *stationary* (translation invariant) random function, whereas the time-scale behavior revealed by the wavelet transform of f is, under all circumstances, at least a limited measure of the time-scale behavior prevailing in f . Thus, comparing the *utility* of the wavelet transform (which is linear) with that of $V\{f(t, \omega)\}$ (which is nonlinear) is unwarranted. Nonlinear transformations often destroy even the Gaussian character of random signals. Linear transformations do not. The statement in Ref. 1 that "at each t , $V\{f(t, \omega)\}$ is an instantaneous Fourier transform" is true, but it's not the Fourier transform of the given f .

The spectrograms produced by Lang and Forinash (Ref. 1, Figs. 1–4) show good frequency and time localization be-

cause the analysis wavelet they chose could be readily matched to the familiar pitch and known frequency of the signal. In this sense, the wavelet transform is indeed quite useful. In measuring and characterizing signals encountered in nature, though, the analyst must ascertain the unknown pitch and frequency content by implementing a suitable signal analysis.

Recall that *pitch* is a rather subjective quantity which, for a pure tone of constant intensity, becomes higher as the frequency increases; but for a pure tone of constant frequency, it becomes lower as the intensity increases. In a signal whose intensity and frequency content are unknown and perhaps are even changing with time (nonstationary), an ideal analysis technique would be able to distinguish whether a decrease in pitch is due to a constant intensity and decreasing frequency or due to a constant frequency and increasing intensity. It should also be able to indicate whether a constant pitch actually results from compensating changes in both frequency and intensity.

In mathematical terminology, the above limitations on wavelet analysis result because the wavelet set is *incomplete* (Ref. 5). In other words, there is a scale component in the signal being analyzed which is orthogonal to each of the wavelets in the set. In the analysis of Ref. 2, the peculiar component is a linear trend, and the French-hat wavelet is the analysis tool. This result is analogous to a theorem in Fourier analysis due to Lerch (Ref. 6) which establishes that if two functions differ by at most a *null* function, then the two functions have the same Fourier transform. A null function is a function whose integral over the domain of interest is zero. The null function intended here is the product of an analysis wavelet and its peculiar orthogonal scale component. Mallat (Ref. 7), though, reports that *completeness* by itself is not enough. Wavelet representations must also be *stable*, meaning small modifications in the signal being analyzed should correspond to small perturbations in the wavelet representation.

To elaborate further, wavelets have been described as mathematical microscopes (Ref. 8). This is indeed a very useful and illustrative analogy, but keep in mind that simple optical lenses suffer from *spherical aberration*. This focusing error results because simple lenses, unlike the human eye, cannot be made with a variable focal length. Variable focal length is what endows the human eye with its unmatched ability to lift hidden images from seemingly meaningless backgrounds. With a simple lens, on the other hand, only the central portion of the lens produces a clear image. The effect becomes even more limiting if the lens is used to take close-up images, since it must then be very convex.

Analogously, wavelet transforms provide a technique for focusing on information of a given scale size, say L , in a random signal of duration T , where $L < T$, and ferreting this information from the signal. The information obtained, however, depends on the analyzing wavelet (lens) used. Farge (Ref. 9) explains the “wavelet aberration” property in this way: “wavelet coefficients combine information about both the signal and the wavelet.” What’s being computed with a wavelet transform is, in fact, an integral of the product of the frequency content of the signal itself and the frequency content of the wavelet [cf. Eq. (8) of Ref. 2]. This effect is not unlike that faced by every experimentalist, who must ask herself “what effect does the instrument I use have on the phenomenon I’m trying to measure?”

Wavelets and wavelet analysis should, however, eventually find their proper place as advanced research tools for signal analysis when we establish which wavelets are best suited for which analyses (*adaptive* wavelet analysis). Reference 1 suggests Meyer (Ref. 10) for answers to the question of which wavelet is appropriate for a particular application. Likewise, we address (Refs. 11 and 12) a specific problem amenable to wavelets, although not like the one considered in Ref. 1.

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Response to “Comment on ‘Time-frequency analysis with the continuous wavelet transform’” [Am. J. Phys. 67 (10), 934–935 (1999)]

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We agree whole-heartedly with Treviño and Andreas in Ref. 1 that wavelets, particularly the continuous transform of Morlet and Grossman which we use, cannot provide desired

information in many problems (such as the study of turbulence). We never meant to assert or imply otherwise. In particular, we certainly agree that our analysis would not shed

light on the trend, median, or variance of a random signal. But our analysis in Ref. 2 was never intended to identify these statistical characteristics of a signal—in fact, our original use of our analysis was to identify small shifts of frequency over time in experimental data (produced in investigations of lattice breather mode solitons). Our analysis served this purpose well (see Ref. 3), and we were also delighted that it worked well with other natural signals, such as human singing (where the ascending harmonics of the phrase “do-re-mi” were clearly visible). This is all we were referring to when we asserted that the transform produces spectrograms that show the frequency of sounds or other signals as they change over time.

We also agree with Treviño and Andreas that the wavelet transform spectrogram shows as much information about the wavelet as it does about the signal, and that the choice of wavelets is important; their metaphor of spherical aberration is entirely apt. But we take mild issue with their assertion that good results are possible with our transform only by knowing *a priori* the characteristics of the signal to be analyzed, and matching the wavelet to the signal. The information provided by our wavelet transform is similar to the information provided by an ordinary Fourier transform, and therefore we feel justified in saying it has considerable utility.

We now consider two points in Ref. 1 that we believe need clarification.

First, while we did not intend to suggest that the Wigner transform is not useful for analyzing some phenomena (I. Daubechies⁴ reports that the Wigner transform is of most value for analyzing signals of brief duration), we still believe that it is less satisfactory than the continuous wavelet transform for the analysis we performed. As we mentioned in Ref. 2, the Wigner transform produces artifacts which indicate the presence of energy at times and frequencies where it is reasonable to conclude no energy should be present. In fact, consider a signal that consists of two “notes” of limited duration, one at time t_1 with frequency w_1 , and one at time t_2 with frequency w_2 . The Wigner transform will show an artifact at time $(t_1 + t_2)/2$ and at frequency $(w_1 + w_2)/2$ of amplitude similar to the two notes—even if t_1 and t_2 and w_1 and w_2 are arbitrarily far apart. (See Y. Meyer in Ref. 5.) We would suggest great caution in interpreting these artifacts as nonlinearities in turbulent signals, as Treviño and Andreas do in Ref. 1. We should note, by the way, that we were motivated to discuss the Wigner transform because we became aware that some in the physics community were familiar with the use of the Wigner transform for time-frequency analysis similar to ours, but not with the continuous wavelet transform.

Second, we are somewhat confused by the discussion in Ref. 1 concerning certain points of Fourier analysis. In Ref. 1, the authors state that wavelets form an incomplete set.

Wavelet sets are in fact typically designed to be complete bases, and continuous wavelet transforms are designed to be invertible, so that the original signal can always be recovered from the wavelet expansion or transform. Since the original signal can be recovered from the wavelet transform, there is no information “lost” in the transform, at least in theory. (Here we must mention that the continuous wavelet transform in Ref. 2, using the Morlet wavelet, is not invertible. However, a small perturbation of the wavelet will produce a continuous wavelet transform which is invertible; see Ref. 6 or 7. We used the Morlet wavelet, which is often used in practice, for simplicity.)

Also, in Ref. 1 it is stated that a null function is one whose integral over the domain is 0, and the authors cite the theorem of Lerch that two functions have the same Fourier transform if they differ by a null function. The authors describe products of a wavelet with certain components of a signal as null functions, and they invoke this as an explanation for why wavelet analysis does not show certain kinds of information. But the analogy does not work since a null function is really one which is essentially constantly zero—in fact, a continuous null function will be constantly zero. (Here, by essentially constantly zero we mean a function which differs from zero only on a set of measure zero, such as a finite or countable set. Treviño and Andreas intend that a null function is one whose integral over the domain is zero, but if that is taken to be the definition of a null function, then the theorem of Lerch is *no longer true*, e.g., consider the function $\sin x$ on the domain $[0, 2\pi]$.) Thus we believe that null functions are not the explanation for why the continuous wavelet transform does not display information such as the trend of a signal.

Wavelet analysis has generated intense interest and activity, and it is perhaps inevitable that unreasonable claims would be made about it. It was our hope in writing Ref. 2 to provide an understanding of time-frequency analysis and the use of the continuous wavelet transform in the context of time-frequency analysis that would enable our readers to use this new tool in an appropriate way.

¹G. Treviño and E. L. Andreas, “Comments on ‘Time-frequency analysis with the continuous wavelet transform’ [Am. J. Phys. **66** (9), 794–797 (1998)],” Am. J. Phys. **67** (10), 934–935 (1999).

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