Waves.

These notes are not a substitute for reading the book and working problems.

The wave equation.

Many different types of waves are governed either exactly or approximately by the linear wave equation; $\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2}$ where *y(x,t)* represents the wave function.

- If the equation is describing waves on a string *y(x,t)* represents the vertical displacement of the string from equilibrium at location *x* and time *t*.
- If the equation is describing sound waves *y(x,t)* represents the difference of normal atmospheric pressure from equilibrium at location *x* and time *t* (the pressure squared is proportional to the loudness).
- If the equation is describing water waves *y(x,t)* represents the vertical displacement of the water from equilibrium at location *x* and time *t*.
- If the equation is describing electromagnetic waves *y(x,t)* represents the amount of magnetic or electric field at location *x* and time *t* .
    1. For waves on a string the linear wave equation can be derived from (and is equivalent to) Newton's second law, $F = ma$ (recall that *a* is the second time derivative of location which is what is on the right side of the wave equation). The speed of the wave in this case is $v = \sqrt{T/\rho}$ where *T* is the tension in the string and ρ is the density per length.
    2. Maxwell started with Maxwell's equations and derived the following equation for the y component of the electric field: $\frac{\partial^2 E_y(x,t)}{\partial x^2} = \varepsilon_o \mu_o \frac{\partial^2 E_y(x,t)}{\partial t^2}$. He found a similar equation for the magnetic field. If we compare this with the wave equation we see that it *is* the wave equation provided these electromagnetic waves travel at a speed of $v = \frac{1}{\sqrt{\varepsilon_o \mu_o}} = c = 3\times 10^8 m/s$. When Maxwell figured this out he was the first person to realize that light (and radio, x-rays, uv, ir, etc.) were electromagnetic waves.

Wave functions (solutions to the wave equation).

Almost any function which has the form $y(x,t) = f(x \pm vt)$ will be a solution to the wave equation. However it is most common to focus on sine and cosine functions because other shapes can be constructed by adding together sines and cosines: $y(x,t) = y_o \cos(kx - \omega t + \phi)$. Here the maximum amplitude (height) of the wave is y_o, the wave number in *1/m* is *k*, the angular frequency ω in *rad/s* is and ϕ is the phase angle in radians. For electromagnetic waves we have $E(x,t) = E_o \cos(kx - \omega t + \phi)$ and

$B(x,t) = B_o \cos(kx - \omega t + \phi)$ where E_o and B_o are the maximum electric and magnetic fields.

The following relationships are useful for all types of waves:

- $\omega = 2\pi f$ where f is the frequency of oscillation in Hertz (Hz).
  - Frequency tells us the pitch of sound waves or the color of electromagnetic waves.
- $k = 2\pi / \lambda$ where λ is the wavelength in meters.
- $v = f\lambda = \omega / k$ is the speed of the wave.
- The wave function $y(x,t) = y_o \cos(kx - \omega t + \phi)$ describes a wave traveling to the right. Changing the minus sign to a plus sign describes a wave traveling to the left.
- Keep in mind that the material or fields making up the waves don't really go any where, they just oscillate in magnitude in the same place. Water waves sloshes up and down, string vibrates up and down, air in a sound waves oscillates back and forth and the electric and magnetic fields change strength and direction but the air, water, string and fields don't actually go anywhere.
- The velocity of the material (water, string, air) as it moves back and forth or up and down is different from the wave velocity. The material velocity is given by $v(x,t) = \frac{\partial y(x,t)}{\partial t} = y_o \omega \sin(kx - \omega t + \phi)$. So the maximum velocity of a point on the string (air molecule, water drop) is $v_{\max} = y_o \omega$ where y_o is the maximum amplitude.
- The acceleration of the material (water, string, air) as it moves back and forth or up and down is given by the second derivative of location: $a(x,t) = \frac{\partial^2 y(x,t)}{\partial t^2} = -y_o \omega^2 \cos(kx - \omega t + \phi)$. So the maximum acceleration of a drop of water (piece of the string, etc.) is $a_{\max} = y_o \omega^2$.
- Breaking waves like you see at the beach *do* travel but they are not described by the linear wave equation or sines and cosines.

How do we know that $y(x,t) = y_o \cos(kx - \omega t + \phi)$ is the correct wave function for the wave equation? One way to tell is to substitute it directly into the wave equation. The wave equation says two derivatives with respect to *x* on the left side should equal $1/v^2$ times two derivatives with respect to *t* on the right. When you do this you find out everything cancels except $v = \omega / k$ which is where this relationship comes from in the first place.

Adding waves.

Because this is a linear wave equation we are talking about we can add together any number of sines or cosines and still have a solution. So if $y_1(x,t)$ and $y_2(x,t)$ are both solutions then $y_1(x,t) + y_2(x,t)$ is also a solution also (you can easily prove this by substituting $y(x,t) = y_1(x,t) + y_2(x,t)$ into the wave equation and see what happens). Here are some physical phenomena that can be explained by adding waves (check your book for explanations and equations which govern these phenomena):

- **Beats**: the frequency heard when two waves are present with slightly different frequencies is $f = f_1 - f_2$ where f_1 and f_2 are the original frequencies.
- **Standing waves**: two identical waves traveling in opposite directions will produce a standing vibration like the vibration of a guitar string.
- **Path difference**: Two identical waves which travel different distances may arrive at the same location either in phase, out of phase or partially in/out of phase. If they are in phase they add (**constructive interference**) if they are out of phase they destroy each other (**destructive interference**). The concept of path difference explains the following:
    - Young's double slit experiment.
    - Iridescence (colors on peacock wings, moth wings, oyster shells).
    - Colors on a compact disk (CD).
    - Colors on soap bubbles and other thin films.
    - Sound cancellation headphones.
- **Diffraction**: Waves will interact with objects and openings which are close to the same size as the wavelength of the wave. In general this causes the path of the wave to deviate from a straight line direction.
    - Single slit diffraction is an example.
    - All optical instruments (including radio telescopes, microscopes, telescopes, cameras etc.) have limited resolution because of diffraction. The smallest angle by which two objects can be apart and still be distinguished is given by $\theta_R = 1.22\frac{\lambda}{D}$ where D is the diameter of the opening and λ is the wavelength of light (radio wave, etc.) being used.
- **Fourier's theorem**:
    - Any periodic shape (triangle wave, square wave, etc.) can be formed by adding together a combination of sines and cosines.
    - Any periodic shape (triangle wave, square wave, etc.)can be broken down into a combination of sines and cosines.
    - We can hear the difference between a trumpet and a clarinet even if they play the same note because, although the fundamental frequency is the same, there are other frequencies (other sines or cosines) present which change the sound.
    - A musical synthesizer adds the appropriate sines and cosines to get the wave function to look (and sound) like that of a clarinet or trumpet.

Doppler shift.

The frequencies of both light and sound undergo a shift if either the source of the wave or the observer of the wave is moving.

- For sound the shift is $f = f'\left(\frac{v + v_o}{v - v_s}\right)$ where v is the speed of sound, v_s is the speed of the source and v_o is the speed of the observer. These speeds are positive numbers if the object and source are getting closer together but become negative if they are moving away from each other.
- For light the shift is $f = f'\left(\frac{c + v_s}{c - v_s}\right)^{1/2}$ where c is the speed of light and v_s is the relative speed between observer and source. The same sign conventions apply.
- Applications of the Doppler shift include:
    - Police radar.
    - Weather radar.
    - Measurement of the motion and rotation of stars and galaxies.
    - Measurements of the expansion rate of the universe.
- It is important to realize that applications of the Doppler shift (police radar, Doppler weather radar, etc.) are *not* measuring the time needed for the wave to return. Police radar and weather radar bounce microwaves off of objects and measure the change in frequency $\Delta f = f - f'$. Given the speed of light and the frequency shift the speed of the object can be calculated from the Doppler shift equation.