NOTES 9: **Harmonic Motion and Chaos.**

- These are supplementary notes only, they do not take the place of reading the text book.
- Like learning to ride a bicycle, you can only learn physics by practicing. There are worked examples in the book, homework problems, problems worked in class, and problems worked in the student study guide to help you practice.

Key concepts:

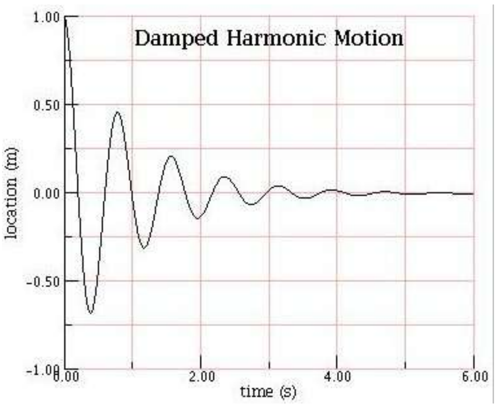
1. The motion of a mass hanging from a spring, a pendulum swinging through small angles and many other motions can be described by a single type of equation. A motion is called **harmonic motion** if the force acting on the object can be written as a constant times the displacement (a Hooke's law force): $F = -kx$ where k is a constant.
2. Since it is also true that $F = ma$ and $a = d^2x/dt^2$ we have the following version of Newton's second law for a mass on a spring called the **simple harmonic oscillator** equation: $-k x = m\, d^2x/dt^2$. This is called a differential equation because it has derivatives in it. the location of the mass, x, is now no longer just a number, it is a function of time (the location of the mass on the spring changes with time).
3. The equation $-kx = md^2x/dt^2$ describes the forces acting and the acceleration but it does not describe the motion of the object. Solutions to this equation describe the actual motion and can be written in the form $x(t) = A \cos(\omega t + \phi)$ where A is the maximum **amplitude**, ω the **angular frequency** and ϕ is the **phase** (sine and exponential functions are also solutions but we will use mostly cosine). The solution says that as time advances the location, x(t), of the mass on the spring goes from some maximum positive value, A, to the minimum value of -A because the maximum and minimum values of cos are +1 and -1. That motion repeats as t continues to get larger. In other words the mass bobs up and down at the end of the spring.
4. The angular frequency, ω, in radians per second is related to the number of cycles (from max to min and back to max) per second, f, by $\omega = 2\pi f$. f is called the **frequency** of oscillation and is measured in Hertz where 1 Hz= 1 cylce/sec. A higher frequency (or angular frequency) means the mass bobs up and down faster.
5. The period of the motion in seconds is the inverse of frequency: $T = 1/f$.
6. The phase angle, ϕ , tells us when the clock started on our measurements. If $\phi = 0$ then we started the motion at t = 0 when the mass was at the top of its oscillation because cos (0) = 1. If $\phi = \pi/2$ radians it means we started the clock with t = 0 at the midpoint of the oscillation because $\cos(\pi/2) = 0$.
7. A first derivative of x(t) gives the velocity of the mass: $v(t) = dx(t)/dt = -A\omega \sin(\omega t + \phi)$. A second derivative of x(t) gives the acceleration: $a(t) = d^2x(t)/dt^2 = -A\omega^2\cos(\omega t + \phi)$. So the maximum velocity of the mass is $A\omega$ and the maximum acceleration is $A\omega^2$. Notice that the maximum velocity occurs as the mass passes through the equilibrium point (x = 0) but acceleration is a maximum the same time the amplitude is a maximum.
8. We can prove that $x(t) = A \cos(\omega t + \phi)$ is a solution to $-k\, x(t) = m\, d^2x(t)/dt^2$ by direct substitution. Substituting $x(t) = A \cos(\omega t + \phi)$ on the left side and the second derivative (from 7) on the right gives $-kA \cos(\omega t + \phi) = -mA\omega^2\cos(\omega t + \phi)$. Dividing the cos out leaves $-kA = -mA\omega^2$ or $\omega = (k/m)^{1/2}$. This tells us that the rate of oscillation, ω, is determined by the mass and the spring constant, k. A stiffer spring vibrates faster, a smaller mass vibrates faster.
9. $\omega = (k/m)^{1/2}$ is called the **natural frequency** of the oscillator and is sometimes written as ω_o.

What if there is friction?

1. We can add a velocity dependent friction term to Newton's law for a harmonic oscillator to get the **damped harmonic oscillator** equation: $-k x - b\, dx/dt =$

m d^2x/dt^2 (this is still F = ma but now with two forces acting on the mass, the spring and velocity dependent friction, -b dx/dt).

2. Solutions to this equation look like x(t) = A exp(-bt/2m) cos (ωt + ϕ). Notice the only difference between this solution and the harmonic oscillator above is the exponential function multiplying the amplitude, A. What does this mean? It means the amplitude decreases over time. This is as we expect; friction would have the effect of gradually making the oscillation smaller.



3. The rate at which the oscillation slows (how fast the exponential decreases) is determined by b, the friction coefficient. If the exponential stops the motion before even one whole oscillation the motion is said to be over damped. If it gets through exactly one oscillation it is said to be critically damped. If it gets through one or more oscillations it is under damped.
4. How do we know x(t) = A exp(-bt/2m) cos (ωt + ϕ) is a solution? Direct substitution into -k x - b dx/dt = m d^2x/dt^2 ! We take two derivatives of x(t) and put it on the right side, one derivative times b on the left and then k times x(t) on the left. If you do that (it requires a bit of math) you find out that $\omega = (k/m - (b/2m)^2)^{1/2}$. Notice that the oscillator would vibrate with the natural frequency, $\omega_o = (k/m)^{1/2}$, if the friction were zero (b = 0).

What if we don't want the oscillator to stop?

1. We can add one more force to the left side of Newton's law called a driving force. The simplest kind of driving force is a sinusoidal force F_o sin (ωt). Notice that the frequency, ω, here is the input frequency that you drive the oscillator, not the natural frequency that the oscillator wants to vibrate at.
2. Here is the equation for the **driven, damped harmonic oscillator**: F_o sin (ωt) -k x - b dx/dt = m d^2x/dt^2 again, all the forces acting on the mass are on the left side, mass times acceleration on the right.
3. What do the solutions look like? Surprise! x(t) = A cos (ωt + ϕ) once again. Only now ω is <u>not</u> the natural frequency, it is the driving frequency. The driving force compensates for the friction and the mass keeps oscillating.
4. How do we prove this is a solution? Direct substitution! Put in x(t) every where there is an x in the equation F_o sin (ωt) -k x - b dx/dt = m d^2x/dt^2. What we find in this case when the algebra is done is that $A=\frac{F_o/m}{\sqrt{(\omega^2-\omega_o^2)^2+(\omega b/m)^2}}$ where $\omega_o = (k/m)^{1/2}$. The amplitude depends on the driving force amplitude, F_o, the mass, the friction, the driving frequency and the natural frequency. In doing this we also find out that $\phi=\tan^{-1}(\frac{\omega_o b/m}{\omega^2-\omega_o^2})$. In other words the effect of driving the oscillator is that its maximum amplitude (starting point of the clock) occurs at a different time.
5. The natural frequency, $\omega_o = (k/m)^{1/2}$, is fixed by the spring constant, k, and the mass but the driving frequency, ω, is your choice, you can shove the mass at any rate you wish. This means there is one special case for the amplitude. If

your driving frequency, ω, happens to equal the natural frequency, ω_o, you get the maximum value for A. This is called **resonance** and that particular driving frequency which causes the largest amplitude is called the **resonant frequency**. (Think of pushing your kid brother on a swing. The swing wants to go back and forth at its natural frequency. If you push it at the same frequency as the natural frequency the swings get larger. If you push at some other frequency the swings do not get larger.). More examples below.

Applications and examples done in class, on quizzes, etc:

Harmonic oscillators and resonance turn up nearly everywhere.

1. Almost anything that vibrates (pendulums, atoms in a solid, cork bobbing on water, etc.) can be modeled as a harmonic oscillator, at least in a first approximation.
2. A common example of resonance is your car's suspension system. If your tire is out of balance it will act as a periodic driving force on the spring and shock absorbing system. What you notice is that if you drive at a certain speed you get a big vibration but driving slower or faster gives a smaller vibration. When you feel the largest vibration the wheel is driving the system at the natural frequency and you are at resonance.
3. Sound from an acoustic guitar does not come from the strings. The strings drive the top and bottom of the guitar and the vibrating surfaces cause the air to move in a way that we call sound. A good guitar is one that will resonate well with the strings, meaning that it has natural frequencies in the same range as those of the string.
4. Big bass speakers can often cause things to vibrate if the sound is at just the right frequency. The speakers are driving the objects that vibrate at their resonant frequency.
5. The Tacoma narrows bridge collapsed because it had a natural frequency close to the driving frequency of the wind. (Ask your instructor to show the film clip.)
6. Magnetic resonance imaging, electron spin resonance and many other techniques for investigation the atomic structure of solids depend on resonance. Using an oscillating magnetic or electric field as the driver it is possible to get the magnetic spin of an electron or other subatomic particle to resonate (flip 180^o in sync with the driver) if the driving frequency matches the natural frequency. This allows you to determine the natural frequency of the electron. This in turn tells you something about the environment of the electron, it will have a different natural frequency depending on the kind of chemical bonds there is with neighboring atoms.

Supplementary material

Pendulums.

1. Suppose we suspend a mass, m, from a string of length l and pull it back by a small angle θ. The rotational equivalent to Newton's law for circular motion is $\tau = I\alpha$.
2. In this case the torque, τ, provided by the weight acting at a distance l from the pivot and the angle θ between the force and the lever arm is $\tau = mgl \sin \theta$.
3. The moment of inertia, I for a mass rotating around the suspension point is $I = ml^2$.
4. Angular acceleration is second derivative of the angular position, $\alpha = d^2 \theta/dt^2$.
5. So Newton's law becomes $-mgl \sin \theta = I\, d^2 \theta/dt^2$.
6. This is not a harmonic oscillator because of the sine function. For small angles (in radians) however $\sin(\theta) \approx \theta$ (try this on your calculator) and we can write $-mgl\, \theta = I\, d^2 \theta/dt^2$. Using $I = ml^2$ and dividing out the mass gives $-g\theta = ld^2 \theta/dt^2$.
7. This is the identical equation to the harmonic oscillator, only the letters have been changed. This tell us that the solution has to be the same and looks like θ

θ(t) = θ_o cos (ωt + ϕ) with maximum amplitude, θ_o and natural frequency, ω_o = $(g/l)^{1/2}$.

8. Notice that the frequency of the pendulum is independent of the mass which is why pendulums are used as the timing mechanism in clocks.
9. Notice also that we could use a very sensitive pendulum to measure the gravitational constant, g = 9.8 m/s^2 by measuring slight changes in the frequency.

Chaos.

Lets add one more force, - qx^3 to the damped, driven harmonic oscillator: $F_o \sin(\omega t) - k x - b\, dx/dt - q x^3 = md^2x/dt^2$ where q is a constant. This new term is called nonlinear because, unlike -k x, it does not graph as a straight line. As you will see in the chaos computer exercise, it causes the mass on a spring to have some very interesting behavior. There is no closed form solution to this equation (we can't find a formula for x(t)) but it can be solved numerically on a computer. The resulting behavior is not random because we can determine exactly where the mass will go at each time step. For some choices of the parameters k, m, b and q we get normal looking harmonic motion. But for other choices of the parameters k, m, b and q we get behavior that does not seem to repeat in any obvious pattern. Since the motion is determined by an equation it is not random, although it may appear pretty random. In these cases we see that it is also true that if we start the mass from two initial locations which are close together the behavior quickly becomes quite different. A deterministic system (not random because we can solve the equations) which is very sensitive to initial conditions is said to be **chaotic**.

An example of a chaotic system is the weather. Although we can write down and solve (by computer) the equations for determining the weather next week based on todays weather as input the equations are chaotic. This means that if we use slightly different input parameters (temperature, pressure, etc.) for todays weather we may get very different results for our prediction. This is why weather predictions are not very accurate for several days in advance, the predictions are too dependent on the input parameters. Even with the best data we have for input we could still be off by a little and that slight uncertainty in our initial data makes the prediction unreliable.