NOTES 6: **Momentum.**

- These are supplementary notes only, they do not take the place of reading the text book.
- Like learning to ride a bicycle, you can only learn physics by practicing. There are worked examples in the book, homework problems, problems worked in class, and problems worked in the student study guide to help you practice.

Key concepts:

1. **Momentum** is a vector: $\mathbf{p} = m\mathbf{v}$.
2. Note that, although both momentum and kinetic energy have mass and velocity in them they are <u>not</u> the same thing. Momentum is a vector and contains information about direction but kinetic energy ($\frac{1}{2} mv^2$) does not.
3. The fact that momentum is a vector means that a ball hitting a wall and bouncing back with the same velocity has a change in momentum of 2p. For example if a ball of mass 0.2kg comes approaches a wall with a velocity of 10m/s and reflects with a velocity of -10m/s then the change in momentum is $\Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_f$ = 0.2kg (10m/s)$\mathbf{i}$ - 0.2kg (-10m/s)$\mathbf{i}$ = +40$\mathbf{i}$ kgm/s.
4. A force can cause a change in momentum: $\mathbf{F}\Delta t = \Delta\mathbf{p}$ where $\mathbf{F}\Delta t$ is called the impulse and $\Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_f$. Dividing by Δt gives Newton's second law in a different form: $\mathbf{F} = d\mathbf{p}/dt$ since acceleration is the derivative of velocity and mass is a constant (unless you are moving fast enough to invoke special relativity).
5. If the external forces are zero then momentum is conserved: $\mathbf{F}\Delta t = 0 = \Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_f$ so $\mathbf{p}_f = \mathbf{p}_f$. **Conservation of momentum** is our second, great conservation law.
6. So far we have been talking about a single object. All of the above is also true for a group of objects (two, three, four, etc. billiard balls for example) if we use $\mathbf{p}$ to mean the sum of all momentums, $\sum\mathbf{p}$ (kind of like net force in Newton's second law). So it could be that $\mathbf{p}_i = \mathbf{p}_{1i} + \mathbf{p}_{2i} + \mathbf{p}_{3i}$ + ... and likewise for $\mathbf{p}_f$. So really we should write the conservation of momentum as $\sum\mathbf{p}_i = \sum\mathbf{p}_f$.
7. Keep in mind that these are vectors which are shorthand for separate x, y and z equations. So for momentum conservation we have: $p_{1ix} + p_{2ix} + p_{3ix}$ + ... = $p_{1fx} + p_{2fx} + p_{3fx}$ + ... and $p_{1iy} + p_{2iy}$ + ... = $p_{1fy} + p_{2fy}$ + ... etc.
8. Internal forces have no effect on conservation of momentum because they always occur in action/reaction pairs. The net effect for the system as a whole is zero (for example, taking your hand hand pushing on your head does not make you move, even if you are on a frictionless surface). This is why friction played no role in the ballistic pendulum lab, the forces between the ball and the catcher were internal forces to the collision.

1. There is an angular equivalent to momentum called **angular momentum**: $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. (Note that this is the vector cross product – the second way of multiplying vectors. $\mathbf{L}$ is a vector whose direction is determined by the right hand rule.)
2. A second definition of angular momentum is equivalent to the first: $\mathbf{L} = I\omega$ where I is the moment of inertia and ω is the angular velocity. Generally the first definition works best for single masses and the second for rotating objects (disks, balls, etc.).
3. Our third great conservation law is **conservation of angular momentum**: $\sum\mathbf{L}_i = \sum\mathbf{L}_f$ if external torques are zero.
4. Not only the amount of angular momentum is conserved but also the direction. This is why a top does not turn over, it wants to maintain the axis of rotation in the same direction, conserving the amount and direction of the angular momentum.

Applications and examples done in class, on quizzes, etc:

1. We get to talk about car crashes and other types of collisions using momentum conservation. There are two kinds of collisions, elastic and inelastic.
   - In an **elastic collision** the objects bounce off of each other and no energy is converted into work (heat etc.). In this case the kinetic energy is the same before and after the collision.
   - In an **inelastic collision** some kinetic energy is converted into other forms (so kinetic energy is not the same after the collision as it was before). Total energy is, of course, always conserved.
   - Momentum is conserved in both types of collisions if no external forces (gravity, etc.) act.
2. Atoms of a gas colliding with the walls of a container conserve their momentum, thus applying a force to the walls of the container. This will provide us with a microscopic definition of pressure. Since temperature is a measure of the average kinetic energy of the molecules we expect that as the temperature increases both the kinetic energy and therefore the momentum increase. This gives us part of the ideal gas law: Pressure in a closed container increases with temperature.
3. Many atomic and sub atomic particles have an intrinsic angular momentum called **spin** (for example elections, neutrons and protons each have a spin of ½). Angular momentum conservation is a core concept in magnetic and electron spin resonance imaging.

Supplementary material

Because internal forces all cancel (as far as conservation of momentum is concerned) it is helpful to define something called the **center of mass**. Objects will act according to Newton's laws as if all the mass is located at the center of mass (with possibly some rotation around the center of mass). This includes objects that come apart. So the center of mass of a bomb follows the same projectile motion trajectory whether or not the bomb explodes. The x- component of the center of mass is defined to be: $x_{cm}=\frac{\int x\,dm}{\int dm}$ where dm is the amount of mass located at a distance x from the y-axis and the integral adds up all the pieces until you have the whole object (the integral can become a sum if the pieces are discrete and have mass m_i: $x_{cm}=\frac{\sum x_i m_i}{\sum m_i}$). Similar definitions give the y location of the center of mass, y_{cm}.

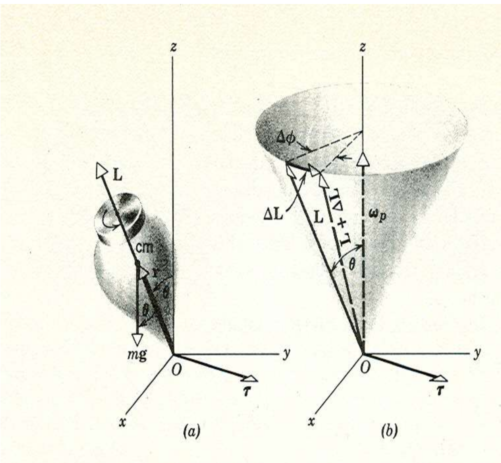
The rocket problem. Conservation of momentum can be used to solve the problem of a rocket which is burning fuel (and thus losing mass). Conservation of momentum has the momentum of the system of rocket plus fuel before and after firing to be equal: $p_i = p_f$ or $mV = (m + \Delta m)(V + \Delta V) - \Delta m (V - v)$ where the term on the left is the rocket plus fuel initially. The first term on the right is the rocket minus a little bit of fuel Δm (so Δm is a negative number) and a new speed of V + ΔV. The second term on the right is the fuel's momentum, ejected in the other direction with a speed v relative to the rocket. If we multiply this out we get: $mV = mV + V\Delta m + m\Delta V + \Delta m\Delta V - \Delta mV + v\Delta m$
We can drop the ΔmΔV term because these will be small when we let the deltas become infinitesimals. That leaves: $0 = m\Delta V + v\Delta m$ or $\Delta V = - v\Delta m/m$ (again, Δm is a negative number so the change in velocity is actually positive). Now we integrate both sides from a full rocket with mass M (empty rocket) + m_o (fuel) to M (empty rocket): $V - V_o = -v\int_{m=M+m_o}^{m=M}\frac{dm}{m} = v\ln(1-\frac{m_o}{M})$ which gives the change in the velocity of a rocket of mass (when empty) of M which

burns a mass of fuel of mass m_o which leaves the rocket with velocity v.
Note that the change in momentum (impulse) is $\Delta(p_f - p_i) = m\Delta V + v\Delta m$ or since the change in momentum is the impulse we have

$$\frac{\Delta p}{\Delta t} = \sum F_{ext} = m\frac{dV}{dt} + v\frac{dm}{dt}$$

where the last term is called the thrust. Notice that if the rocket has an initial velocity (and momentum) of zero the final momentum of the system of rocket and fuel must also be zero. The rocket goes one way, the fuel the opposite direction. The center of mass of the system remains in the same location as the starting point.

Precession of a top. If you watch a top spinning you will notice that it does not fall over but it also does not keep its axis exactly in the same direction. The axis of rotation slowly traces out a circle around a vertical axis. This motion is called precession and is the result of the effect of gravity (an external torque) acting on the center of mass.
Consider a top spinning at an angle θ from the vertical. The angular momentum vector **L** points along the axis. The point of contact of the top with the ground acts as a pivot. Gravity acts down at the center of mass which is at a location **r** from the pivot. Using the right hand rule, the torque produced by the gravitational force is perpendicular to the angular momentum, **L** and is horizontal. The relationship between torque, the definition of torque and angular momentum motion is: $\tau = \mathbf{r} \times \mathbf{F} = I\alpha = \frac{d\mathbf{L}}{dt}$ (this is the form of Newton's law for circular analogous to **F** = d**p**/dt above). From this we see that any change in **L** (d**L**) is in the same direction as the torque. Since the cross product of the gravitational force with the lever arm gives a torque which is horizontal, the change in angular momentum is also horizontal, thus the axis of rotation circles around the vertical.



The precessional frequency (how fast it goes around the vertical, not the rotational frequency) is given by: $\omega_p = mgr/L$